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**Noise-induced changes to the
bifurcation behaviour of semi-
implicit Euler methods for stochastic
delay differential equations**

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Noise-induced changes to the bifurcation behaviour of semi-implicit Euler methods for stochastic delay differential equations

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Stewart J Norton

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Abstract

We are concerned with estimating parameter values at which bifurcations occur in stochastic delay differential equations. After a brief review of bifurcation, we employ a numerical approach and consider how bifurcation values are influenced by the choice of numerical scheme and the step length and by the level of white noise present in the equation. In this paper we provide a formulaic relationship between the estimated bifurcation value, the level of noise, the choice of numerical scheme and the step length. We are able to show that in the presence of noise there may be some loss of order in the accuracy of the approximation to the true bifurcation value compared to the use of the same approach in the absence of noise.

Keywords: stochastic delay equations, bifurcations, numerical methods

AMS subject classification: 34K18,65C30,65P30

1 Introduction

For a parameter-dependent differential equation, the concept of a bifurcation value is familiar (see, for example, [15]): it is the value of a parameter at which there is a fundamental change in the qualitative behaviour of the set of possible solutions. In general there are several types of bifurcation, but in the case of simple linear delay differential equations,

$$y'(t) = \lambda y(t - 1) \tag{1}$$

one can make the concept explicit through considering changes in sign of the characteristic values.

Definition 1.1 *For equation (1) the characteristic values are the roots of*

$$a = \lambda e^{-a}; \tag{2}$$

for such a root, the corresponding function e^{at} is a characteristic function.

It can be shown that the characteristic roots are isolated. It follows that solutions to (1) are formed as linear combinations of the (infinitely many) characteristic functions obtained in this way and the asymptotic behaviour of any given solution (as $t \rightarrow \infty$) will be determined by the dominant (with greatest real part) characteristic root present in the particular solution. As the parameter λ varies, the characteristic roots will vary, and of key interest is the value $\lambda = -\frac{\pi}{2}$ which is the value at which the dominant characteristic root has zero real part. This is a very important bifurcation value because for $\lambda > -\frac{\pi}{2}$

all the characteristic functions tend to zero as $t \rightarrow \infty$ while for $\lambda < -\frac{\pi}{2}$ the dominant characteristic function grows without bound.

For nonlinear equations, such as the delay logistic equation,

$$y'(t) = \lambda y(t-1)(1-y(t)) \quad (3)$$

one proceeds to analyse a linearised version about one or other of the steady states $y(t) = 0, 1$. In the case of linearisation about $y(t) = 0$ one obtains the linear equation (1) and thus one can gain insights into bifurcations of (3) by studying (1). However, it is important to realise that for the linear equation, the bifurcation value is where the behaviour of solutions will change suddenly; in the case of nonlinear equations, the bifurcation value is actually the point where the linearisation breaks down, and one needs to study higher order approximations to understand exactly what will happen.

We can make this precise in the following way:

Definition 1.2 *A bifurcation value of the parameter λ arises for equation (1) (respectively, (3)) whenever (2) is satisfied by at least one value of a such that $\Re(a) = 0$.*

Remark 1.1 *Notice that as λ passes through the value $-\frac{\pi}{2}$ we can expect to be able to see the effect of the bifurcation in the graphs of the solution. Therefore we can not only prove that there is a bifurcation, but we can also detect the bifurcation phenomenologically from the graphs.*

On application of a simple numerical scheme to (1) or (3) one obtains a discrete dynamical system and one can seek a discrete bifurcation through analysis- see [10, 11, 12, 13, 23]. Of course, in the case of the nonlinear equation, one obtains a nonlinear discrete system which is analysed through consideration of a linearised version. One would expect that numerical methods applied to a delay equation would approximate the exact bifurcation behaviour. This turns out to be correct and the results can be summarised:

Theorem 1.1 *Apply a strongly stable linear multi-step method of order p with fixed step length $h > 0$ to (1) (respectively (3)). The resulting discrete system has a bifurcation value at $\tilde{\lambda} \approx \lambda = -\frac{\pi}{2}$ and the approximation is $\mathcal{O}(h^p)$ as $h \rightarrow 0$.*

This theorem provides the theoretical backing for a phenomenological approach to detecting bifurcation (see, for example [9]) in which we can detect approximate bifurcation values of the parameter by detecting changes in the long term behaviour of graphs of approximate solutions to the underlying equation.

In this paper we focus on the linear stochastic delay differential equation with instantaneous multiplicative noise.

$$\begin{aligned} dY(t) &= \lambda Y(t-1)dt + \mu Y(t)dW(t), & t \geq 0 \\ Y(t) &= t + \frac{1}{2}, & t \in [-1, 0]. \end{aligned} \quad (4)$$

where the drift term is based on equation (1). As we have already seen, this is a prototype for understanding various nonlinear problems too.

The case $\mu = 0$ is covered by our previous discussion. Our aim is to ascertain how bifurcations may be affected by the value of μ , by the choice of numerical method and by the choice of step length $h > 0$. We would like to be able to make a statement similar to Theorem 1.1 which relates the choice of method, size (governed by μ) of the noise term and the step length h to the bifurcation value. In the later sections of this paper, we are able to provide reliable conclusions of this type.

As we have already explained, for deterministic problems, one can establish through analysis the parameter values where bifurcations occur, and this can often be backed up using experimental evidence based on graphs produced using numerical approximations. In the case of stochastic problems, it would be convenient to use simulated solutions as a basis for detection. It is quite difficult to make the a precise definition of a phenomenon that is detected visually. As an attempt to be reasonably precise, while retaining the sense that judgement plays an important role, we define the concept of a phenomenological or P-bifurcation:

Definition 1.3 *A P-bifurcation occurs if the stationary measure of a random process changes its shape. It is detected through observing changes in the graph of simulated solutions.*

In [14, 22] we showed the weakness of P-bifurcations to give a clear indication of the bifurcations. Fine judgements were difficult to make, and one was forced to conclude that there were ranges of parameter values rather than single fixed values where changes in behaviour might occur.

A much more precise analysis is possible, in principle, if we use the definition of a Dynamical or D-bifurcation, which uses the concept of Lyapunov exponents – a close analogue in the stochastic case of the characteristic values we discussed earlier.

Definition 1.4 *Dynamical or D-bifurcations are phenomena in families (ψ_α) of random dynamical systems which are related to sign changes of Lyapunov exponents $\mathcal{L}_i(\mu_\lambda)$ of ψ_λ -invariant measures μ_λ , where λ is a bifurcation parameter. See [1]; also chapter 9 of [2].*

A linear stochastic delay equation has infinitely many Lyapunov exponents (see [8]) and for our approach we are interested in the principal (right most) Lyapunov exponent(s) in the complex plane. In other words, we are concerned to detect the Lyapunov exponents that are closely related to the principal characteristic values we discussed in the deterministic case.

Definition 1.5 *The principal Lyapunov exponent is defined as*

$$\Lambda = \lim_{t \rightarrow \infty} \sup E\left(\frac{1}{t} \log |Y(t)|\right). \quad (5)$$

For simplicity, we use the semi-implicit Euler methods (also known as the stochastic θ -methods or θ – Maruyama methods) described in [16] to solve equation (4), leading to the numerical schemes

$$y_{n+1} = y_n + (1 - \theta)h\lambda y_{n-N} + \theta h\lambda y_{n+1-N} + \mu y_n \Delta W_n, \quad (6)$$

where $Nh = 1$ and y_{-N}, \dots, y_0 are given by our initial function. We use $\theta = 0, 0.5, 1$. Note that $\theta = 0$ gives the classical Euler-Maruyama method. We draw attention to the fact that, for $\theta \neq 0$, there would be a natural diffusion-implicit stochastic analogue of the classical θ -methods. However it is easy to show that such a method diverges with probability unity and this has led to the stochastic θ -methods being defined in the above way. For more information on numerical schemes and their behaviour for stochastic equations see [4, 5, 6, 7, 16, 18, 19, 20, 21].

For the bifurcation analysis, we shall be calculating an approximation to the principal Lyapunov exponent. We return to this in the next section.

Remark 1.2 *One might question whether the value of the principal Lyapunov exponent will depend on the choice of initial function. It is known in the deterministic case that, depending upon the initial function, certain characteristic (eigen-)functions may be missing from the expansion of the particular solution. However, for stochastic problems, the presence of noise will ensure that the expression (5) will give, with probability 1, a fixed value for the principal Lyapunov function.*

2 Methodology and simple experimental results

For a range of values of λ over an interval containing $-\frac{\pi}{2}$ we used Matlab to simulate a large number of solution trajectories of our equation over the large interval $[0, T]$ for fixed values of μ, θ and step size h . We calculate $S = \sup_{[T-\epsilon, T]}(|Y(t)|)$ for each solution trajectory and calculate $L = \frac{\log(S)}{T}$ which might be taken as an estimate for the (local) Lyapunov exponent. We can now estimate the probability distribution of the values of L that we have found.

2.1 Brief overview of experimental results

Our previous work showed that taking $T = 5000$ and $\epsilon = 5$ give us consistent results on each trajectory for calculating S . For each λ , 500 trajectories were simulated and the values of L were recorded. Histograms of the 500 values of L for each value of h, μ, θ consistently produced typical bell shaped distributions. Sample histograms are shown in figure 1 for $h = 0.1, \mu = 0.1, \theta = 0$ and for values of $\lambda = -1.49$, close to the bifurcation value suggested for the deterministic equation, and values either side of this. We note that at the value of $\lambda = -1.49$ close to the bifurcation value the interval of values of L straddle zero, while for the value $\lambda = -1.34$ and $\lambda = -1.65$, where we would expect the solutions to all converge and diverge respectively, we get all negative and then all positive values for L . This is entirely expected for our search for D-bifurcations. The Kolmogorov Smirnov test is a standard statistical test that can be applied to a dataset to gain evidence on the distribution of this dataset. We used SPSS, a commercial statistical tool, and found statistically, using the Kolmogorov Smirnov test, that none of our datasets were significantly different from a normal distribution for each value of θ and λ .

We also tabulated the minimum, mean, maximum and standard deviation of L . Table 1 shows a sample of these values, to 6 decimal places, for $h = 0.1, \mu = 0.1, \theta = 0$. (We actually found the values to 14 decimal places and for the 51 values of λ from -1.80 to -1.30

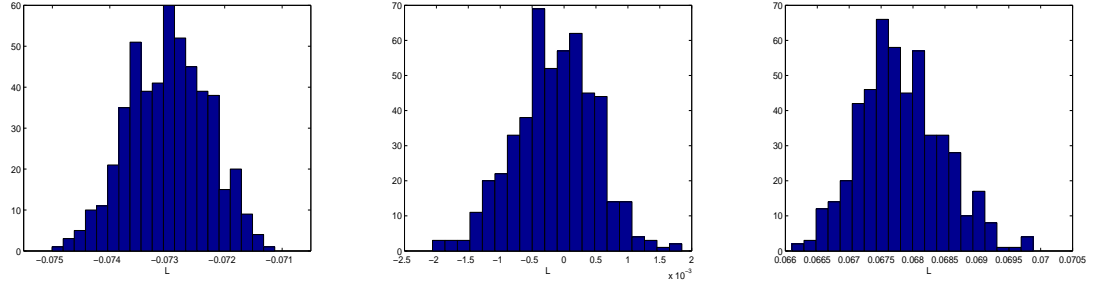


Figure 1: Histogram of the 500 values of L for $\theta = 0$, $\mu = 0.1$ and stepsize $h = 0.1$, using fixed Brownian paths.

Left: $\lambda = -1.34$

Middle: $\lambda = -1.49$

Right: $\lambda = -1.65$

in steps of 0.01). We present these results in detail because they arise from experiments that are not exactly repeatable in order that the results of our analysis are independently verifiable.

2.2 Conclusions based on simple experiments

Based on the experimental results described above we can draw some conclusions. We present these as Conjectures, because no mathematical *proof* is available. However the statistical evidence that we have gathered for them provides a strong scientific basis for drawing these conclusions.

Conjecture 2.1 *For each fixed value of λ, μ, θ, h the distribution of L is normal.*

It follows immediately that

Corollary 2.1 *For each fixed value of λ, μ, θ, h the distribution of L_{mean} is normal.*

We can also combine the results of these experiments with those of our earlier paper [22] to draw the following conclusion:

Conjecture 2.2 *For fixed μ, θ and h , L_{mean} comes from a normal distribution whose mean is accurately represented in the form*

$$\alpha\lambda^2 + \beta\lambda + \gamma$$

3 Further analysis and results

We already know that for a fixed μ and h we obtain an excellent fit for L_{mean} as a quadratic function of λ . We can now investigate the fit for L_{mean} if we fix λ and μ . We need to begin by determining an appropriate model to choose. We base this on the following insight: For the deterministic case we know that it can be shown (theoretically) that the numerical bifurcation point approximates the exact bifurcation to the order of the method.

λ	L_{\min}	L_{mean}	L_{\max}	$L_{\text{Standard deviation}}$
-1.80	0.125696	0.127203	0.129467	0.000636
-1.78	0.117992	0.119548	0.121777	0.000577
-1.76	0.110093	0.111871	0.113710	0.000607
-1.74	0.102413	0.104070	0.105727	0.000623
-1.72	0.094544	0.096156	0.097865	0.000572
-1.70	0.086109	0.088214	0.090290	0.000607
-1.68	0.078722	0.080134	0.081711	0.000559
-1.66	0.068938	0.071973	0.073805	0.000628
-1.64	0.061837	0.063713	0.065542	0.000618
-1.62	0.053654	0.055401	0.057312	0.000611
-1.60	0.045156	0.046968	0.048924	0.000636
-1.58	0.036722	0.038417	0.040472	0.000639
-1.56	0.027892	0.029741	0.032026	0.000658
-1.54	0.018852	0.021045	0.022701	0.000633
-1.52	0.010178	0.012110	0.014359	0.000628
-1.50	0.001550	0.003210	0.005319	0.000643
-1.48	-0.007740	-0.005871	-0.003872	0.000643
-1.46	-0.016888	-0.015106	-0.013391	0.000633
-1.44	-0.026084	-0.024429	-0.022442	0.000663
-1.42	-0.035916	-0.033894	-0.031982	0.000686
-1.40	-0.045154	-0.043385	-0.041458	0.000654
-1.38	-0.054813	-0.053148	-0.051360	0.000642
-1.36	-0.065142	-0.062943	-0.060652	0.000658
-1.34	-0.074742	-0.072934	-0.070458	0.000684
-1.32	-0.085634	-0.083084	-0.081031	0.000650
-1.30	-0.095153	-0.093354	-0.090911	0.000666

Table 1: Summary of L values for $h = 0.1, \mu = 0.1, \theta = 0$ for a sample of the values used for λ

Equation	R
$L_{\text{mean}} = 0.4795h^2 - 0.00074$.954
$L_{\text{mean}} = 0.2641\sqrt{h} - 0.07847$.995
$L_{\text{mean}} = 0.2835h - 0.02506$.998
$L_{\text{mean}} = 0.170250h + 0.107215\sqrt{h} - 0.047085$.99993
$L_{\text{mean}} = -0.143577h^2 + 0.362442h - 0.031696$.999997
$L_{\text{mean}} = -0.1215886h^2 + 0.3323459h + 0.0170465\sqrt{h} - 0.0341816$	1
$L_{\text{mean}} = -0.2140995h^{1.5} + 0.4660743h - 0.0180839\sqrt{h} - 0.0309053$	1

Table 2: Simple regression formulae for $\theta = 0$ and $\lambda = -1.50$

Therefore it makes sense to base our models on the order of the numerical methods in use. It has been shown in [17] that the Euler Maruyama method has strong order of convergence $\gamma = 0.5$ and weak order of convergence $\gamma = 1$. Consequently we looked for a relationship using $h^{\frac{1}{2}}$ and h as the dominant terms. The results of experiments with different combinations in the models are given in table 2 for the case $\lambda = -1.50$, together with the correlation coefficients, R . The closer the value of R is to 1, the better the fit.

The conclusions here need to be interpreted with care and there is scope for further experimentation to reach a completely firm conclusion. One must bear in mind the fact that, by introducing additional complexity in the model, one may obtain falsely accurate results. Both of the final two equations provide an almost perfect fit of the data points ($R = 1$ to 10 significant figures) but there is a much stronger dependency on the terms in h and of higher order than on the term in \sqrt{h} . Both this observation, and further experimentation with other values of λ has led us to conclude that we should use the quadratic model in our analysis but this decision is provisional and needs to be reviewed when further analytical and/or numerical evidence becomes available.

Conjecture 3.1 *For fixed λ, θ, μ the mean of the distribution for L_{mean} depends on h according to a model of the form*

$$\alpha h^2 + \beta h + \gamma$$

Based on Conjectures 2.2 and 3.1 we can calculate the coefficients for a model that combines (for fixed μ) the two earlier models and provides a direct formula for the mean of the distribution of L_{mean} in terms of λ and h for fixed θ, μ .

4 Results

Tables 3,4,5 list the formulae based on the combined model for the 21 cases we considered. In every case the correlation coefficient $R \approx 1$, indicating an excellent fit of the data. We will use these formulae to derive the approximate D-bifurcation value. In other words, we will find the value of λ in terms of h at the point where $L_{\text{mean}} = 0$, as these will give us a value for λ_{bif} at the bifurcation point.

We can write each equation for $L_{\text{mean}} = 0$ as

$$a\lambda^2 + b\lambda + c + dh + eh^2 = 0 \tag{7}$$

μ	Equation	R
0.00	$L_{\text{mean}} = -.1324985\lambda^2 - .1382667h^2 - .8378776\lambda + .3516786h - .9896782$.999
0.05	$L_{\text{mean}} = -.1323627\lambda^2 - .1386083h^2 - .8372673\lambda + .3520501h - .9888930$.999
0.10	$L_{\text{mean}} = -.1318815\lambda^2 - .1394995h^2 - .8352395\lambda + .3531157h - .9864279$.999
0.15	$L_{\text{mean}} = -.1311747\lambda^2 - .1410140h^2 - .8321481\lambda + .3549305h - .9825467$.999
0.20	$L_{\text{mean}} = -.1301694\lambda^2 - .1433929h^2 - .8277879\lambda + .3576682h - .9771360$.999
0.25	$L_{\text{mean}} = -.1288605\lambda^2 - .1465803h^2 - .8221417\lambda + .3613273h - .9701922$.999
0.30	$L_{\text{mean}} = -.1274131\lambda^2 - .1506332h^2 - .8157371\lambda + .3660038h - .9621623$.999

Table 3: Regression formulae for $\theta = 0$

μ	Equation	R
0.00	$L_{\text{mean}} = -.1385855\lambda^2 - .1337073h^2 - .8835123\lambda - .0019131h - 1.0460225$	1.000
0.05	$L_{\text{mean}} = -.1384194\lambda^2 - .1340008h^2 - .8827959\lambda - .0013900h - 1.0451492$	1.000
0.10	$L_{\text{mean}} = -.1378855\lambda^2 - .1347155h^2 - .8805295\lambda + .0001020h - 1.0424380$	1.000
0.15	$L_{\text{mean}} = -.1371017\lambda^2 - .1360919h^2 - .8770965\lambda + .0027371h - 1.0382274$	1.000
0.20	$L_{\text{mean}} = -.1358635\lambda^2 - .1382031h^2 - .8718543\lambda + .0065807h - 1.0320209$	1.000
0.25	$L_{\text{mean}} = -.1342600\lambda^2 - .1410728h^2 - .8651043\lambda + .0116786h - 1.0240963$	1.000
0.30	$L_{\text{mean}} = -.1323592\lambda^2 - .1450999h^2 - .8570765\lambda + .0183135h - 1.0146818$	1.000

Table 4: Regression formulae for $\theta = 0.5$

μ	Equation	R
0.00	$L_{\text{mean}} = -.1613891\lambda^2 - .3028416h^2 - 1.0160353\lambda - .3373565h - 1.1974603$.997
0.05	$L_{\text{mean}} = -.1610697\lambda^2 - .3023180h^2 - 1.0147701\lambda - .3367636h - 1.1961002$.997
0.10	$L_{\text{mean}} = -.1603796\lambda^2 - .3009093h^2 - 1.0118034\lambda - .3348000h - 1.1926925$.997
0.15	$L_{\text{mean}} = -.1593837\lambda^2 - .2984645h^2 - 1.0073553\lambda - .3315786h - 1.1874168$.997
0.20	$L_{\text{mean}} = -.1577823\lambda^2 - .2951257h^2 - 1.0004679\lambda - .3269734h - 1.1795373$.997
0.25	$L_{\text{mean}} = -.1555875\lambda^2 - .2909900h^2 - 0.9911997\lambda - .3209129h - 1.1691346$.997
0.30	$L_{\text{mean}} = -.1528499\lambda^2 - .2864202h^2 - 0.9797878\lambda - .3131905h - 1.1564873$.997

Table 5: Regression formulae for $\theta = 1$

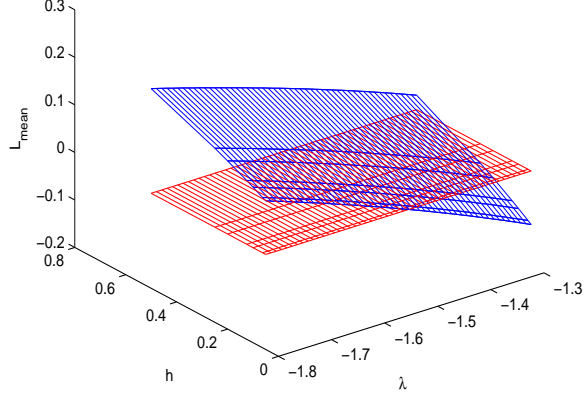


Figure 2: L_{mean} against λ and h for $\theta = 0$ and $\mu = 0.1$, together with the plane $L_{\text{mean}} = 0$

We can solve the equation for λ in terms of increasing powers of h .

First, for convenience, we let $D^2 = b^2 - 4ac$.

Using the quadratic formula, we obtain

$$\begin{aligned}\lambda &= \frac{-b \pm \sqrt{b^2 - 4a(c + dh + eh^2)}}{2a} \\ &= \frac{-b \pm D\sqrt{1 - 4a(dh + eh^2)/D^2}}{2a}\end{aligned}\quad (8)$$

Now, if we have

$$-D^2 \leq 4a(dh + eh^2) \leq D^2 \quad (9)$$

we can expand equation (8) in terms of h . With reference to Figure 2, we take the larger root of equation (8) which becomes (noting a is negative)

$$\begin{aligned}\lambda &= \frac{-b - D[1 - \frac{1}{2}4a(dh + eh^2)/D^2 - \frac{1}{8}16a^2(dh + eh^2)^2/D^4 + \dots]}{2a} \\ &= \frac{(-b - D)}{2a} + \frac{(dh + eh^2)}{D} + \frac{a(dh + eh^2)^2}{D^3} + \dots \\ &= \frac{(-b - D)}{2a} + \frac{d}{D}h + (\frac{e}{D} + \frac{ad^2}{D^3})h^2 + \dots \text{terms in } h^3 \text{ and higher}\end{aligned}\quad (10)$$

If we substitute the values of the coefficients of our equation for $\theta = 0$ and $\mu = 0.10$ we find

$$\lambda = -1.570419 - 0.838713h - 0.551683h^2 + \dots \quad (11)$$

Substituting in equation (9) shows that this expansion is valid for $-0.737 \leq h \leq 3.268$, a range which clearly includes all of the reasonable values of h .

θ	μ	Equation
0	0.00	$\lambda = -1.571912 + 0.834695h - 0.547274h^2$
	0.05	$\lambda = -1.571528 + 0.835740h - 0.548515h^2$
	0.10	$\lambda = -1.570419 + 0.838713h - 0.551683h^2$
	0.15	$\lambda = -1.548589 + 0.843807h - 0.557288h^2$
	0.20	$\lambda = -1.566100 + 0.851446h - 0.566000h^2$
	0.25	$\lambda = -1.562969 + 0.861674h - 0.577721h^2$
	0.30	$\lambda = -1.559247 + 0.874770h - 0.593051h^2$
0.5	0.00	$\lambda = -1.571133 - 0.004270h - 0.298433h^2$
	0.05	$\lambda = -1.570780 - 0.003103h - 0.299150h^2$
	0.10	$\lambda = -1.569733 + 0.000278h - 0.300944h^2$
	0.15	$\lambda = -1.568048 + 0.006121h - 0.304378h^2$
	0.20	$\lambda = -1.565737 + 0.014742h - 0.309660h^2$
	0.25	$\lambda = -1.562846 + 0.026218h - 0.316905h^2$
	0.30	$\lambda = -1.559441 + 0.041722h - 0.327114h^2$
1	0.00	$\lambda = -1.570182 - 0.662504h - 0.733830h^2$
	0.05	$\lambda = -1.569866 - 0.661547h - 0.732357h^2$
	0.10	$\lambda = -1.568979 - 0.658357h - 0.728407h^2$
	0.15	$\lambda = -1.567504 - 0.653117h - 0.721807h^2$
	0.20	$\lambda = -1.565492 - 0.645613h - 0.712586h^2$
	0.25	$\lambda = -1.562970 - 0.635669h - 0.700930h^2$
	0.30	$\lambda = -1.559988 - 0.622769h - 0.687417h^2$

Table 6: Equation for λ in terms of h at $L_{\text{mean}} = 0$

We have repeated this analysis for all 21 of the cases tabled above, and the equations are shown in table 6.

4.1 Conclusions

In line with the results we know already for the deterministic equation, when $\theta = 0, 1$ we obtain formulae for λ_{bif} which is a close $\mathcal{O}(h)$ approximation to $-\frac{\pi}{2}$. For $\theta = 0.5$, which corresponds to the second order trapezium method in the deterministic case, the h coefficients are very small, so we have (to working accuracy) an $\mathcal{O}(h^2)$ approximation to $-\frac{\pi}{2}$. In this case it is also evident that as μ , the noise coefficient, increases, the h coefficient in the formula for λ_{bif} becomes more significant.

5 Analysis of the effect of varying noise levels

We note that, by symmetry of the white noise process, we should expect equation (4) will give us a similar family of solutions to equation

$$\begin{aligned}
dY(t) &= \lambda Y(t-1)dt - \mu Y(t)dW(t), & t \geq 0 \\
Y(t) &= t + \frac{1}{2}, & t \in [-1, 0].
\end{aligned} \tag{12}$$

This indicates that the coefficients in our formulae are very likely to depend only on even powers of μ . Hence, for each θ , it makes sense to plot the graphs of each quadratic equation coefficient from table 6 against μ^2 . Figure 3 shows these graphs for $\theta = 1$. We can calculate the regression formulae for these three coefficients and repeat for the other two θ values. In all nine cases $R = 1.000$, giving near perfect linear fits, and confirming the dependency on μ^2 . Using linear regression on the figures for $\theta = 1$ we obtain

$$\begin{aligned} h^2 \text{ coefficient} &= -0.73357 + 0.51702\mu^2 \\ h \text{ coefficient} &= -0.66280 + 0.43965\mu^2 \\ \text{constant term} &= -1.57011 + 0.11346\mu^2 \\ &\approx -\frac{\pi}{2} + 0.11346\mu^2 \end{aligned}$$

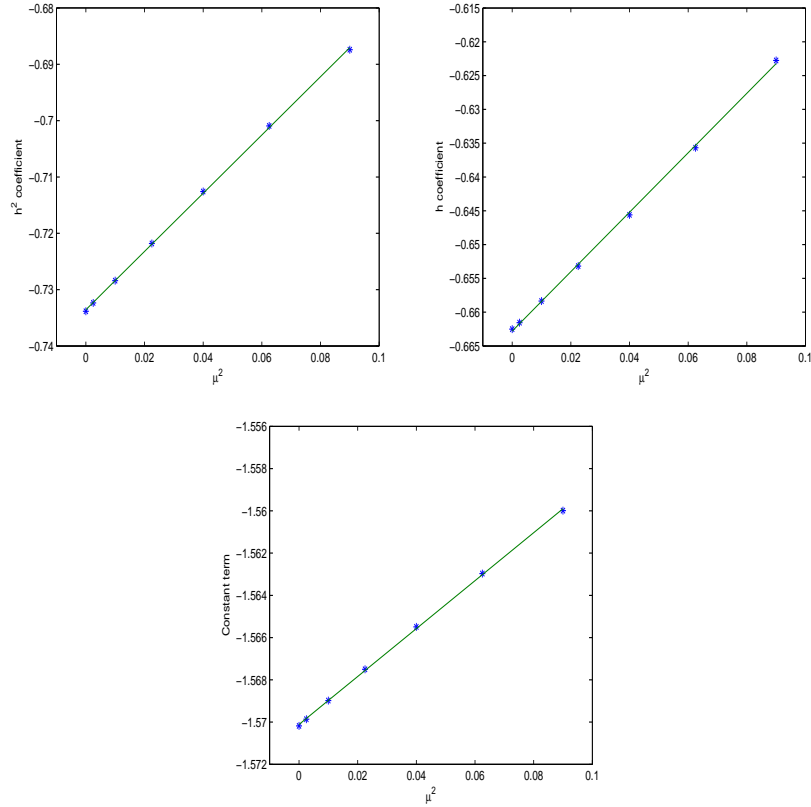


Figure 3: Regression lines for the quadratic coefficients against μ^2 for $\theta = 1$
Top left: h^2 coefficient Top right: h coefficient
Bottom: Constant term

Our original aim was to determine a formulaic relationship between the bifurcation value of the parameter, the choice of method, the step length, and the noise level μ . We are finally in a position to give precisely these formulae:

For $\theta = 0$,

$$\lambda = (-1.57183 + 0.14092\mu^2) + (0.83428 + 0.44343\mu^2)h + (-0.54665 - 0.50542\mu^2)h^2 \quad (13)$$

For $\theta = 0.5$,

$$\lambda = (-1.57105 + 0.13017\mu^2) + (-0.00481 + 0.50704\mu^2)h + (-0.29785 - 0.31503\mu^2)h^2 \quad (14)$$

For $\theta = 1$,

$$\lambda = (-1.57011 + 0.11346\mu^2) + (-0.66280 + 0.43965\mu^2)h + (-0.73357 + 0.51702\mu^2)h^2 \quad (15)$$

These results are very satisfactory because they build in a natural way on the insights we already have.

1. By putting $\mu = 0$, we recover an excellent representation of the known behaviour of these schemes for the deterministic problem.
2. We can observe the way in which the presence of noise influences the approximation of the bifurcation point in each of the methods.
 - (a) For the cases $\theta = 0, 1$ the deterministic problem leads to an $\mathcal{O}(h)$ approximation of the exact bifurcation value. We can see that the presence of noise leads to a change in each of the three coefficients in equations (13) and (15). This means that, in the limit as $h \rightarrow 0$ we would expect to obtain an approximation for the bifurcation value that differs from $-\frac{\pi}{2}$ by an amount proportional to μ^2 . During the limiting process, we expect to observe $\mathcal{O}(h)$ convergence.
 - (b) For the case $\theta = 0.5$ one needs to interpret equation (14) particularly carefully. If μ is small, then (14) will provide an *apparent* $\mathcal{O}(h^2)$ rate of convergence as $h \rightarrow 0$ in experimental data. It is only when the value of μ is larger that the *true* convergence rate $\mathcal{O}(h)$ will become apparent. This explains why some experiments involving equations with small noise can predict an $\mathcal{O}(h^2)$ approximation to λ .

6 Conclusions and further work

The results of this paper provide a systematic approach to analysing the approximate bifurcation values for equation (4) and show how the approximations are influenced both by the choice of numerical scheme and its step length and by the level of noise in the equation. There are several observations and questions that are significant and motivate further investigation:

1. The estimates of the bifurcation value (obtained by putting $h = 0$ in equations (13), (14) and (15)) all indicate that the presence of the noise has induced a change in the bifurcation value. Can this change be established analytically for the underlying SDDE, or is it nevertheless an artefact induced by the numerical scheme?

2. The presence of the μ^2 term in the coefficient of h in (14) means that the observed behaviour of approximations might change in a significant way when the level of noise varies. One needs to be particularly careful in applying *small noise* insights to general problems.
3. Can a formula be developed that combines equations (13), (14) and (15) into a single expression with θ as parameter. Can such an expression lead to establishing some critical value of θ (other than 0, 0.5, 1) for which the numerical approach displays enhanced properties.

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